On the Convergence Speed of Tetration

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Published Online: 15 Oct. 2018

Abstract: In 2011, in his book “La strana coda della serie n^n^n...n”, M. Ripà analyzed some properties involving the rightmost figures of integer tetration, the iterated exponentiation \( b^a \), characterized by an increasing number of stable digits for any base \( a > 1 \). A few conjectures arose about how many new stable digits are generated by any unitary increment of the hyperexponent \( b \), and Ripà indicated this value as \( V(a) \) or “convergence speed” of \( a \). In fact, when \( b \) is large enough, \( V(a) \) seems to not depend from \( b \), taking on a (strictly positive) unique value, and many observations supported this claim. Moreover, we claim that \( V(a) = 1 \) for any \( a \mod 25 \in \{ 2, 3, 4, 6, 8, 9, 11, 12, 13, 14, 16, 17, 19, 21, 22, 23 \} \) and \( V(a) \geq 2 \) otherwise.

Keywords: Number theory, Power tower, Tetration, Chinese reminder theorem, Charmichael function, Euler’s totient function, Exponentiation, Integer sequence, Graham’s number, Convergence speed, Modular arithmetic, Stable digit, Rightmost digit.

2010 Mathematics Subject Classification: 11A07, 11F33.

1 Introduction

In the present paper, we introduce some conjectures involving the rightmost digits of the Tetration \( b^a = a^{a^{a^{...}}} \) \( (b\text{-times}) \) [1], observing that, when the hyperexponent \( b \) is sufficiently large and \( a \mod 25 \not\equiv \{ 0, 1, 5, 7, 10, 15, 18, 20, 24 \} \), the amount of new stable digits generated by any unitary increment of \( b \) is unitary as well: it depends only on the congruence modulo 25 of the base \( a \) [2].

This new result, if formally proved, would contribute to improve big numbers rightmost digits calculations, opening new scenarios in cryptography/cryptanalysis too [3-4].
2 Convergence Speed and Congruence (mod 25)

It is well known that, for any arbitrarily large \( n \), \( b^a \) originates a string of \( n \) stable figures, thus we can say that \( b^a \) is well-defined modulo \( 10^n \), for any \( b \geq b'(n, a) \) [2-5-6].

We can easily prove also that, \( \forall k \in \mathbb{N}, a^{20^k+1} \equiv a \pmod{25} \). In fact, \( \forall n \in \mathbb{N} \setminus \{0\}, \lambda(n) \leq \varphi(n) \). Thus, \( \lambda(25) = \varphi(25) = 20 \).
Let \( a \) be such that \( \gcd(a, 25) = 1 \) \( \iff \gcd(a, 5) = 1 \), \( a^{\varphi(25)} \equiv 1 \pmod{25} \) \( \Rightarrow a^{20+1} \equiv a \pmod{25} \). Hence \( a^{20^k+1} \equiv a \pmod{25} \).
For any \( a \) such that \( a \equiv 5 \pmod{10} \), \( \forall m \in \mathbb{N} \setminus \{0, 1\}, a^m \equiv 0 \pmod{25} \) \( \Rightarrow a^m \equiv a^{m+1} \pmod{25} \). Therefore, \( a^2 \pmod{25} \equiv a^3 \pmod{25} \equiv \ldots \equiv a^{20+1} \pmod{25} \equiv \ldots \equiv a^{20^k+1} \pmod{25} \). \( \square \)

Let us now introduce the definition of “convergence speed” as it was originally presented by Ripà in his book about the rightmost digits of \( b^a \) [1].

**Definition 1**: Let \( a \in \mathbb{N} \setminus \{1\} \) be an arbitrary base which is not a multiple of 10 and let \( b \in \mathbb{N} \setminus \{0, 1\} \) be such that \( (b-1)a \equiv b^a \pmod{10^d} \) \( \land \) \( (b-1)a \not\equiv b^a \pmod{10^{d+1}} \), where \( d \in \mathbb{N} \), we consider \( V(a) \equiv b^a \equiv (b+1)a \pmod{10^{((d+V(a))} \) \( \land \) \( b^a \not\equiv (b+1)a \pmod{10^{(d+V(a)+1)}} \).

For simplicity, from here on out, we refer to \( V(a) \) as the “convergence speed” of the natural base \( 1 < a \not\equiv 0 \pmod{10} \) of the tetration \( (b\geq b')a \).

3 The Conjectures about \( V(a) \)

In this section we present the conjectures and a few remarks to point out their main implications.

**Conjecture 1**: \( \forall a \in \mathbb{N} \setminus \{1, 2\} \) such that \( a \not\equiv 0 \pmod{10} \), \( \exists b' < a \in \mathbb{N} \setminus \{0\} \) \( \exists \), \( \forall b \geq b' \), \( V(a) \in \mathbb{N} \setminus \{0\} \) is constant (see A317905 of the OEIS - ruling out the first term of the sequence [2]).

**Conjecture 2**: Assume \( b \in \mathbb{N} \setminus \{0, 1, 2\} \), \( \forall a \in \mathbb{N} \setminus \{1\} \) such that \( a \not\equiv 0 \pmod{10} \),

\[ b^a \equiv (b+1)a \pmod{10^{(b-2)V(a)}} \).

**Remark**: If Conjecture 2 holds, it follows that \( (b - 2) \cdot V(a) \leq d + V(a) \), hence \( \forall b > 3 \),
\[ V(a) \leq \frac{d}{b-3} \text{ (e.g., if } a = 143^625 \text{ and } b \geq 5, 4 = V(a) \leq \frac{9+6+6+5+25}{b-3} = \frac{17+(b-4)-4}{b-3} \text{ is true).} \]

**Ripà’s hypothesis**: \( \forall a \in \mathbb{N} \setminus \{1, 2\} \) such that \( a \not\equiv 0 \pmod{10} \), \( \exists b' < a \in \mathbb{N} \setminus \{0\} \) \( \exists \), \( \forall b \geq b' \),

\[ \{ V(a) = 1 \iff a \pmod{25} \in \mathbb{C} = \{2, 3, 4, 6, 8, 9, 11, 12, 13, 14, 16, 17, 19, 21, 22, 23\} \}
\[ \{ V(a) \geq 2 \iff a \pmod{25} \in \mathbb{C} = \{0, 1, 5, 7, 15, 18, 24\} \}

Remark: It is very important to notice that, given $a \, (mod\, 25) \in \mathbb{C}$ (or equivalently $V(a) = 1$), it follows that $V(a^m) \geq 2$, $\forall m = 5 \cdot n \in \mathbb{N}\backslash\{0\}$, and $V(a^m) = 1$ otherwise (for any $m$ such that $m \, (mod\, 10) \equiv \{1, 2, 3, 4, 6, 7, 8, 9\}$).

On the contrary, for any base such that $a \, (mod\, 25) \in \mathbb{C}$, $V(a^n) \geq 2$, since $a^n \, (mod\, 25) \in \mathbb{C}$ too ($\forall n \in \mathbb{N}\backslash\{0\}$). We point out that $V(a) \geq 2 \Rightarrow a^{m+1} \, (mod\, 25) \equiv a \, (mod\, 25)$, $\forall m = 4 \cdot n$.

Conjecture 3: $\forall v \in \mathbb{N}\backslash\{0\}$, $\exists a$, not a multiple of 10, such that $V(a) = v$.

Remark: In order to prove Conjecture 3, it is sufficient to verify that, for any $n$-digits long base $a := a_n ... a_2a_1$, where $a_1 = a_2 = \cdots = a_n = 9$, $V(a = 9 ... 9) = n \, (\forall b)$ (see [1], pp. 25-26).

From Ripà’s hypothesis, it follows that $a(n = 1) \in \mathbb{C} \Rightarrow V(a) = 1$ and $a(n \geq 2) \in \mathbb{C} \Rightarrow V(a) \geq 2$.

Conjecture 4: Let $\text{len}(a(i))$ denote the length of the $i$-th term any (strictly positive) integer sequence $a(n)$ constructed through the juxtaposition of integers, $\forall i \in \mathbb{N}$ such that $\text{len}(a(i)) \geq 2$,

$a(i)a(i) \equiv a(i+1)a(i+1)\, (mod\, 10^{\text{len}(a(i))})$.

Remark: This conjecture was firstly introduced in 2011 [1] and two examples of this property are given by the sequences A317903 and A317824 of the OEIS [7-8].

4 Conclusion

It is not easy to provide a short proof of any of the conjectures introduced in Section 3 and this could be the subject of another paper that we hope to release in the near future.

We conclude with a very important question that we wish to answer: “Is it possible to identify a new function $\mathcal{R}(V(a)) := \min_{R}\{|V(a + k \cdot R) - V(a)|, \forall V(a) = n \in \mathbb{N}\backslash\{0\} \text{ and } \forall k \in \mathbb{N}_0$ (e.g., $\mathcal{R}(V(a) = 1) = \mathcal{R}(1) = 25$ by Ripà’s hypothesis)”?.

Any original contribute to help us to prove the aforementioned conjectures would be appreciated.

References


